ELECTIVE COURSE "ELEMENTS OF THE QUALITATIVE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS" FOR BACHELORS OF THE PEDAGOGICAL DIRECTION OF EDUCATION

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Abstract. At present, the problems connected with the development and introduction of various elective courses in the educational process of higher educational institutions (universities), which contribute to the introduction of students to the modern achievements of science and practice, are becoming increasingly important. In the field of mathematical education, when preparing as specialists in the field of mathematics, computer technology, mechanics, and in the preparation of a future secondary school teacher, this problem becomes more acute because of the wide variety of modern scientific theories that claim to be included in curricula, as well as the greater laboriousness such courses. The purpose of this article is to substantiate a possible version of such an elective course for students of the pedagogical direction of education, as well as a description of the structure and content of the course. To develop and substantiate the results obtained in the article, we used the method of comparative analysis, as well as the method of experimental confirmation of theoretical hypotheses. In particular, scientific and methodological sources devoted to the development of elective courses were analyzed, special mathematical literature was analyzed, approbation of the course in several academic groups of students was carried out. This article presents the possible structure and content of the elective course "Elements of a qualitative theory of ordinary differential equations", the advantages of such a course in terms of developing the necessary competencies and cognitive interest of students are indicated. The proposed elective course is aimed at improving the professional and psychological-pedagogical training of future school teachers, within the framework of which their professional competencies are being improved, and progress is being made in professional development. The article is intended for teachers of mathematical disciplines in higher educational institutions. The structure of the article is such that all teachers of higher education, interested in the complex of questions and problems considered in it, can develop their version of an elective course on the theory of ordinary differential equations.

Keywords: elective course, pedagogical direction of education, qualitative theory of ordinary differential equations, homogeneous equations, preparation of bachelors of pedagogical education.

1 Introduction

We are talking about the course "Elements of a Qualitative Theory of Ordinary Differential Equations" for students studying at the Elabuga Institute of the Kazan (Volga) Federal University in the direction of physical and mathematical education.

The modern qualitative theory of ordinary differential equations is a series of extensive mathematical theories that require, for a study even in a minimal volume, significant time and significant mathematical erudition (Ilyashenko, 2006; Nemytskii & Stepanov, 2016; Obolensky, 2006). The purpose of this course is to study some initial information and facts of the qualitative theory of ordinary differential equations, to develop students' ability to determine the basic properties of solutions for classes of relatively simple ordinary differential equations.

As is known, the qualitative theory of ordinary differential equations studies the properties of solutions of differential equations without finding the solutions themselves. We make a quote from (Ilyashenko, 2006; González & Villalobos Antunez, 2016), which gives an idea of the origin and development of this section of mathematics.

"A. Poincare conceived a qualitative theory as an approach to the study of differential equations, not according to the formulas of their solutions - such formulas, as a rule, do not exist - but directly on their right-hand sides. There was a new discipline at the junction of geometry and analysis. As the main goal, A. Poincare called a qualitative study of the three-body problem. However, the natural geometric questions turned out to be nontrivial even for equations in the plane. With them, he began his research.

At present, the geometric theory of differential equations has strongly branched out. Hamiltonian mechanics separated from it, together with a new branch - the KAM theory; a multidimensional theory of dynamical systems, also called differential dynamics; bifurcation theory; holomorphic dynamics, which studies iterations of rational mappings of the Riemann sphere onto itself; equations on surfaces; theory of relaxation oscillations; a qualitative theory of differential equations on the plane, real and complex.

For the most part, these theories study similar questions:

- What is the local behaviour of solutions (near a singular point)?
- What are the global properties of solutions (in the entire phase space and in infinite time)?
- How do these properties get rearranged (bifurcated) in systems that depend on the parameter, when does this parameter change?

These questions are much better studied in the theory of differential equations on the real plane than in other sections; some of them have been studied with almost full completeness".

The foundations of the qualitative theory of ordinary differential equations were laid at the end of the 19th century by outstanding mathematicians A. Poincaré and A. M. Lyapunov. It seems that the special course (elective course), connected with the study of differential equations, can be very useful for bachelor students in the pedagogical direction of education (specialty "Mathematics and Physics", "Mathematics, Computer Science and Computer Science").

The choice of the concrete content of such a course is connected with certain difficulties, first of all, due to the extraordinary variety of various important and interesting sections of the theory of differential equations. As a possible version of such a course, we proposed the course "Elements of the qualitative theory of ordinary differential equations". However, the small amount of lecture hours devoted to the study of educational material puts rather strict requirements for the selection of material. Indeed, the course should be fairly compact (read for one semester), but at the same time have a complete logical structure.

2 Methodology

1. Methods of research.

During the research, the following research methods were used: analysis of normative documents and sources in the field of pedagogy, teaching methods and mathematics (the theory of differential equations), comparative analysis of sources and pedagogical concepts, systematization and generalization of facts and concepts, method of peer reviews, analysis of student performance listeners of elective courses, pedagogical experiment.

2. Experimental research base.

The experiment on the introduction of the elective course "Elements of a qualitative theory of ordinary differential equations" was conducted on the basis of the Elabuga Institute of the Kazan (Volga) Federal University.

3. Stages of research.

The study was conducted in four stages. At the first stage, scientific literature was studied, the present state of the problem under investigation was analyzed in theory and practice; a study plan was developed. At the second stage, the concept of an elective course was developed, its structure was developed, a selection of theoretical (lecture) material and tasks for solving at seminar sessions was conducted, and a fund of evaluation tools was developed.

At the third stage, the course was tested in the academic groups of the Physics and Mathematics Faculty of the Elabuga Institute of the Kazan (Volga) Federal University. At the fourth (final) stage, the results of the study were processed and formalized.

3 Results

We now turn to the description of the structure and content of the variant of the elective course on the theory of ordinary differential equations.

In our opinion, as a source of initial theoretical information and non-standard problems for seminars, one can take the book of Academician I.G. Petrovsky (Petrovskiy, 1984; Nikolaev, 2018). In it (Chapter II "Simplest Differential Equations"), the methods for constructing integral curves and other questions of the qualitative theory for equations with separating variables, homogeneous equations, first-order linear equations, and equations in complete differentials are considered successively. The solution of problems posed in (Petrovskiy, 1984) helps students to see the theory of differential equations in a new perspective, to make sure that even for externally simple equations, non-trivial problems can be posed. The material of the book (Petrovskiy, 1984) is the first part of the course.

For a deeper understanding of the material, a sufficiently informative (but at the same time familiar and simple) object of study is required. We consider that as such an object we can propose a homogeneous equation of general form

$$y' = f\left(\frac{y}{x}\right)\cdot$$

An effective method for constructing integral curves (and phase portraits) of equations of the form (1) was proposed in (Shilov, 1950; Lyagina, 1951). In Lyagina, an exhaustive analysis of the behaviour of the integral curves of equation

$$y' = \frac{ax^2 + bxy + cy^2}{dx^2 + exy + fy^2}.$$

For this equation, a detailed classification of possible types of behaviour of integral curves is given. The study of this method and the solution of the corresponding problems constitute the second part of the special course.

In addition, students must learn to solve additionally a number of problems characterizing the properties of homogeneous differential equations and illustrating methods for constructing integral curves.

We believe that the variety of problems arising in the theory of ordinary differential equations, as well as the effectiveness of methods of qualitative theory in solving these problems, are demonstrated using the example of homogeneous differential equations studied on a special course. Familiarization of students with such a course causes the students cognitive interest in various aspects of the modern theory of ordinary differential equations.

Let's give examples of mathematical problems solved within the elective course.

Task 1

Prove that the integral curves of equation

$$x^2\frac{dy}{dx} = \frac{y^2}{2} - \sqrt{5x^2 + y^2 + x^2y^2}$$

cross the line $y = 2x$ at an $\frac{\pi}{4}$ angle (Samoilenko et al, 2006).

Solution

For a homogeneous equation $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, it is easy to calculate the tangent of the angle at which the integral curves intersect the ray $y = kx$. Let M be the point of intersection of some integral curve with the straight line $y = kx$ and β is the value of the angle between the tangent drawn to the integral curve at the M point and the axis of abscissae. Then the angle φ between the tangent to the integral curve and the straight line $y = kx$ equals $\beta - \alpha$; thus

$$tg\varphi = tg(\beta - \alpha) = \frac{tg\beta - tg\alpha}{1 + tg\beta \cdot tg\alpha}.$$

The $M\ (x_0 y_0)$ point lies on a straight line $y = kx$; consequently,

$$tg\beta = \left.\frac{dy}{dx}\right|_M = f\left(\frac{y_0}{x_0}\right) = f(k)\cdot$$

Thus, $tg\varphi = \frac{f(k) - k}{1 + kf(k)}$. We write the initial equation in the form

$$\frac{dy}{dx} = \frac{y^2}{2x^2} - \sqrt{5 + \left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^2}.$$

For this equation

$$f\left(\frac{y}{x}\right) = \frac{1}{2}\left(\frac{y}{x}\right)^2 - \sqrt{5 + \left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^2}.$$

So, $f(k) = \frac{k^2}{2} - \sqrt{5 + k^4 + k^2}$. The tangent of the angle of intersection of the integral curves of the original equation with the straight line $y = 2x$ is calculated by the derived formula:

$$tg\varphi = \frac{f(k) - k}{1 + kf(k)} = \frac{f(2) - 2}{1 + 2f(2)} = \frac{-3 - 2}{1 + 2(-3)} = 1\cdot$$

Thus, $\varphi = \pi / 4$, QED.

Task 2

Construct approximately the integral curves of equation (Samoilenko et al, 2006)

$$xy\frac{dy}{dx} + x^2 = 2y^2$$

not solving it.

Solution

In solving the previous problem, a formula was derived for determining the tangent of the angle between the ray $y = kx$ and the integral curve of the homogeneous equation $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ that intersects it. This formula is used to approximate the construction of integral curves of a homogeneous equation.

Since the integral curves of the homogeneous equation intersect the ray $y = kx$ at the same angle, then, by examining the sign of the equation $\frac{f(k)-k}{1+kf(k)}$ as a function of k, one can approximately determine the behaviour of the integral curves of the equation $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

We write down the exercise equation in the form $\frac{dy}{dx} = \frac{2y}{x} - \frac{x}{y}$; receive $f\left(\frac{y}{x}\right) = 2\frac{y}{x} - \frac{x}{y}$. The tangent of the angle of intersection of the integral curve with the ray $y = kx$ is given by

$$tg\varphi = \frac{f(k)-k}{1+kf(k)} = \frac{k-\frac{1}{k}}{1+k\left(2k-\frac{1}{k}\right)} = \frac{k^2-1}{2k^3}.$$

The original equation does not change if x is replaced by $(-x)$ or $(-y)$. Thus, the integral curves must be located symmetrically with respect to the abscissa and ordinate axes. Therefore, it is sufficient to construct them in the first quadrant of the coordinate system, i.e. to investigate formula (7) only for $k > 0$. For the indicated values k $tg\varphi > 0$ if $k > 1$ and $tg\varphi < 0$ if $0 < k < 1$, and when $k \to 0$ $tg\varphi \to -\infty$; then the integral curves intersect the abscissa axis at a right angle.

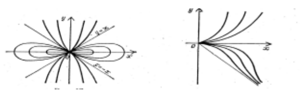

Fig. 1. Several more integral curves, Fig. 2. The integral curves of the initial equation

If $k = 1$, then $tg\varphi = 0$; so the ray $y = x$, $x > 0$ is an integral curve. Having considered several more k values, we obtain sufficient information for an approximate construction of the integral curves of the original equation (Fig. 1).

Task 3

Construct approximately the integral curves of equation (Samoilenko et al, 2006)

$$x\frac{dy}{dx} = y + \sqrt{y^2 + \frac{y^3}{x}}$$

not solving it.

Solution

This equation is determined by $\frac{y}{x} + 1 \geq 0$.

Consider this equation when $x > 0$, rewriting it in the form

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{y^2}{x^2} + \frac{y^3}{x^3}}.$$

In this case

$$f\left(\frac{y}{x}\right) = \frac{y}{x} + \sqrt{\frac{y^2}{x^2} + \frac{y^3}{x^3}},\ f(k) = k + \sqrt{k^2 + k^3}.$$

Therefore, the integral curves intersect the ray $y = kx$, $x > 0$, at an angle φ for which

$$tg\varphi = \frac{|k|\sqrt{1+k}}{1+k^2+k\sqrt{k^2+k^3}}$$

It can be seen from this formula that the rays $y = 0$ and $y = -x$, $x > 0$ (when $k = 0$ and $k = -1$) are integral curves of the initial equation. Investigating the function

$$g(k) = \frac{|k|\sqrt{1+k}}{1+k^2+k\sqrt{k^2+k^3}}$$

on the intervals (-1,0) and (0, + ∞), it is not difficult to construct other integral curves of the initial equation (Fig. 2).

Let $x < 0$, then

$$f\left(\frac{y}{x}\right) = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} + \frac{y^3}{x^3}},\ f(k) = k - \sqrt{k^2 + k^3},$$

and the integral curves intersect the ray $y = kx$, $x < 0$, at an angle φ such that

$$tg\varphi = \frac{-|k|\sqrt{1+k}}{1+k^2-k\sqrt{k^2+k^3}}$$

The rays $y = 0$ and $y = -x$, $(x < 0)$ are the integral curves of the original equation. We construct the remaining integral curves by investigating the function

$$g(k) = \frac{-|k|\sqrt{1+k}}{1+k^2 - k\sqrt{1+k}}$$

on the intervals (-1, 0) and (0, + ∞).

Note that the expression in the denominator of the $g(k)$ function can vanish when $k > 0$:

$$1+k^2 = k^2\sqrt{1+k},\ 1+2k^2+k^4 = k^4+k^5,$$
$$k^5 - 2k^2 - 1 = 0.$$

Investigating the function $z = k^5 - 2k^2 - 1$ (or graphically), it is easy to establish that the equation $k^5 - 2k^2 - 1 = 0$ has one positive root $k = k_0$. Consequently, the ray $y = k_0 x$, $x < 0$ the integral curves of the original equation intersect at a right angle. Calculating the value of the function $g(k)$ at several points $k \in (-1, 0) \cup (0, +\infty)$ if necessary, we obtain sufficient information for approximate construction of the integral curves of the original equation for $x < 0$ (Fig. 3). The behaviour of the integral curves on the entire plane is shown in Fig. 4.

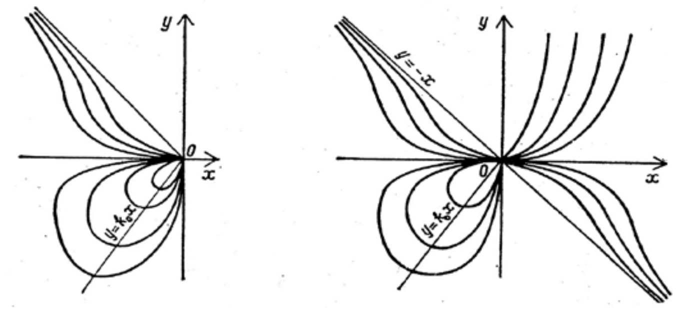

Fig. 3. information to make an approximate integral curve of the original equation, Fig. 4. The behavior of the integral curve in the whole plane

4 Discussion

Recently, issues related to interactive methods of teaching mathematics to university students (Rodionov et al, 2017; Zeytun et al, 2017; Ohly et al, 2017; Lü et al, 2011) have been actively studied. It should be noted that the work directly related to the development of mathematical elective courses is rare. In most works devoted to the problems of modern higher education, the problems of constructing the structure of the course and the selection of a specific material that would significantly increase students' cognitive interest and motivation to improve mathematical competencies are not touched at all.

5 Summary

It is established that the course developed by us corresponds to the level of mathematical culture and theoretical preparation of senior students of the university, it allows successfully to teach students a new section of mathematics for them - a qualitative theory of ordinary differential equations, as well as new methods for solving mathematical problems. It is shown that students are shown and fixed subsequently stable cognitive interest in the theory of differential equations, many students begin to address the search and research problems in this field.

6 Conclusions

The authors believe that the information contained in the article presented by them can be useful both theoretically and in practice for teachers of mathematical disciplines at universities and technical higher educational institutions.

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