METHOD FOR THE EXISTENCE OF THE SOLUTION OF SPATIAL NONLINEAR BOUNDARY VALUE PROBLEMS IN THE THEORY OF ELASTICITY

[a]GULNARA A. YAKUPOVA

Kazan Federal University, 18 Kremlyovskaya street, Kazan 420008, Russia
email: [a]math8@mail.ru

Abstract. The paper studies the solvability of nonlinear boundary value problems of the three-dimensional theory of elasticity for an isotropic inhomogeneous hemisphere under kinematic boundary conditions. We have proved the existence theorem and provided analytical, numerical methods for finding solutions. The development of mathematical methods to investigate the solvability of nonlinear spatial boundary value problems for isotropic and anisotropic inhomogeneous elastic bodies is very relevant. Therefore, the aim of this paper is to prove the existence theorem for solutions for an isotropic inhomogeneous hemisphere under set kinematic boundary conditions. The proposed research method includes reducing the original system of equilibrium equations defined by integral representations for displacements, based on Laplace's fundamental solutions, to a system of three-dimensional singular integral equations, the solvability of which is established using the symbol of the singular operator and the compressed-map principle.

Keywords: inhomogeneous hemisphere, existence theorem, three-dimensional singular integral equations.

1 Introduction

To date, there are not many works devoted to the study of the solvability of spatial boundary value problems raised the elasticity theory (Vorovich, 1989; Novozhilov 1948]. The results have been obtained only for linear boundary value problems and by such well-known methods as the variational method and the method of potential and integral equations, which are based on fundamental solutions of homogeneous equilibrium equations (Novozhilov 1948). By the variational method, the authors of (Vorovich 1989; Mikhlin 1962) solved linear problems for anisotropic elastic bodies in energy spaces. As for equations with constant and piecewise constant coefficients describing the equilibrium state of isotropic and piecewise homogeneous elastic bodies, the fundamental solutions have already been constructed so far. This work aims to study nonlinear boundary value problems for an isotropic inhomogeneous elastic hemisphere. Therefore, we proposed a method that suggests reducing the initial system of equilibrium equations to three-dimensional singular nonlinear integral equations with respect to the auxiliary vector function.

2 Methods

The basis of the method for spatial nonlinear boundary value problems is integral representations for the components of displacements based on fundamental solutions of Laplace's equation. Here, they are constructed using an approach based on the use of the harmonic Green's function of the Dirichlet problem for elastic bodies of a special configuration (ball, half-space, cylinder, etc.) and the theory of harmonic potential for arbitrary elastic bodies. This approach, unlike other proposed methods (Vorovich, 1989; Novozhilov 1948), does not require knowledge of particular solutions of the original homogeneous equations system and allows one to study nonlinear boundary value problems for a wider class of equilibrium equations with variable coefficients. A similar approach was previously used by an isotropic inhomogeneous elastic ellipsoid and a ball under kinematic boundary conditions (Timergalyev 2014; Yakupova 2018). To study the solvability of the system of integral equations, the theory of multidimensional integral equations developed by Professor Mikhlin S.G. is used (Mikhlin, 1962).

3 Results And Discussion

In domain V, which is occupied by the elastic body, a system of equations of form is considered

$$\sigma^{kj}_{,j} + f_k + X_k = 0,\ k = 1,2,3$$

$$\partial V: x_1^2 + x_2^2 + x_3^2 = 1; \tag{1}$$

(hereinafter, a summation from 1 to 3 is carried out with repeated Latin indexes), in which notations are accepted:

$$f_1 = \frac{\partial}{\partial x_j}(\sigma^{j3}\omega_2 - \sigma^{j2}\omega_3),\ f_2 = \frac{\partial}{\partial x_j}(\sigma^{j1}\omega_3 - \sigma^{j3}\omega_1),$$

$$f_3 = \frac{\partial}{\partial x_j}(\sigma^{j2}\omega_1 - \sigma^{j1}\omega_2);$$

$$\sigma^{jj} = 2\mu\varepsilon_{jj} + \lambda\varepsilon,\ \sigma^{jk} \equiv \sigma^{kj} = \mu\varepsilon_{jk},\ j \neq k;\ \varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33};$$

$$\varepsilon_{jk} = e_{jk} + æ_{jk}, e_{jj} = u_{j,j}, e_{jk} = u_{j,k} + u_{k,j}, æ_{jj} = (\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_j^2)/2,$$

$$æ_{jk} = -\omega_j\omega_k, j \neq k, k = 1,2,3; \omega_1 = (u_{3,2} - u_{2,3})/2,$$

$$\omega_2 = (u_{1,3} - u_{3,1})/2,\ \omega_3 = (u_{2,1} - u_{1,2})/2;\ \mu = \frac{E}{2(1+\nu)},$$

$$\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}; \tag{2}$$

hereinafter, symbol $a_{,j}$ stands for partial derivative $a_{,j} = \partial a/\partial x_j$.

The system of equations (1) together with relations (2) describes the equilibrium state of an elastic isotropic inhomogeneous body. In this case: σ^{kj}- components of stresses, ε_{jk} - components of strains, $u = (u_1, u_2, u_3)$ - a vector of displacements, $X_k(k = 1,2,3$ - components of volumetric external forces acting on an elastic body; μ - a shear modulus of elasticity, λ - a Lame's parameter, $E = E(x)$ - a tensile modulus of elasticity, $\nu = \nu(x)$ - a Poisson's ratio, $x = (x_1, x_2, x_3)$ - a point of an elastic body.

If in a system (1) stresses and strains are replaced by expressions from (2), then we obtain a system of equations of equilibrium in displacements:

$$\Delta u_k + \theta_{,k}/(1-2\nu) + l_k(u) + g_k(u) + X_k/\mu = 0, k = 1,2,3, \tag{3}$$

where $l_k(u) = [\mu_{,k}e_{kk} + \mu_{,j}e_{kj} + \lambda_{,k}(e_{11} + e_{22} + e_{33})]/\mu$,

$$g_k(u) = \frac{1}{\mu}\left\{f_k(u) + \frac{\partial}{\partial x_k}[(\mu+\lambda)(æ_{11} + æ_{22} + æ_{33})] + \frac{\partial}{\partial x_j}(\mu æ_{jk})\right\}, \tag{4}$$

$\theta = div\, u,\ \Delta$ - Laplace operator.

It is noteworthy that in the case of linear problems $g_k(u) \equiv 0, k = 1,2,3$.

In addition, if the body is homogeneous, then $l_k(u) \equiv 0, k = 1,2,3$.

Задача A. It is required to find the solution $u = (u_1, u_2, u_3)$ for a system (3) in a hemisphere $V: x_1^2 + x_2^2 + x_3^2 \leq R^2$ $(x_3 \geq 0)$ satisfying a condition ∂V on its boundary

$$u = 0. \tag{5}$$

We will study Problem A in a generalized formulation. Let the following conditions be satisfied: $a)\ E(x),\ \nu(x) \in W_p^{(1)}(V), p > 3;\ b) X_k \in L_p(V), p > 3,\ k = 1,2,3.$

Definition. We will name a generalized solution to Problem A the displacement vector. $u = (u_1, u_2, u_3) \in W_p^{(2)}(V), p > 3,$ *almost everywhere (a.e.), there is (3) satisfying the system and a boundary condition (5).*

Here is $W_p^{(j)}(V)(j = 1,2)$ Sobolev space. By virtue of embedding theorems, there is a generalized solution $u \in C_\alpha^1(\underline{V})$, a $E(x)$, $\nu(x) \in C_\alpha(\underline{V}), \alpha = (p-3)/p$ for Sobolev spaces $W_p^{(j)}(V)$ c $p > 3$.

The solution to problem A will be sought in the form

$$u(x) = \iiint_V G(y,x)\rho(y)dy, \quad dy = dy_1 dy_2 dy_3, \tag{6}$$

where $\rho = (\rho_1, \rho_2, \rho_3)$ - an arbitrary vector function belonging to space $L_p(V), p > 3$; $G(y,x)$ - harmonic Green's function of the Dirichlet problem, which in the case of hemisphere V has the form (Mikhlin, 1962).

$$G(y,x) = \frac{1}{4\pi|y-x|} - \frac{R}{4\pi|y||y^*-x|} - \frac{1}{4\pi\left|\underline{y}-x\right|} + \frac{R}{4\pi|y|\left|\underline{y}^*-x\right|},$$

$y^* = (y_1^*, y_2^*, y_3^*) = R^2 y/|y|^2$ - a point symmetric to a point $y = (y_1, y_2, y_3) \in V$ with respect to a sphere $x_1^2 + x_2^2 + x_3^2 = R^2$; $\underline{y} = (y_1, y_2, -y_3)$ and $\underline{y}^* = (y_1^*, y_2^*, -y_3^*)$ - points symmetric to points $y = (y_1, y_2, y_3) \in V$ and $y^* = (y_1^*, y_2^*, y_3^*) \in V_1: x_1^2 + x_2^2 + x_3^2 \geq R^2, x_3 \geq 0$ with respect to a surface $x_3 = 0$; $\underline{y} = (y_1, y_2, -y_3) \in V_2: x_1^2 + x_2^2 + x_3^2 \leq R^2, x_3 \leq 0; \underline{y}^* = (y_1^*, y_2^*, -y_3^*) \in V_3: x_1^2 + x_2^2 + x_3^2 \geq R^2, x_3 \leq 0;$ $|y-x| = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2 + (y_3-x_3)^2}$.

We will find derivatives up to the second order inclusive of the function $u(x)$. By direct differentiation under the integral sign in (6), we derive

$$\frac{\partial u}{\partial x_j} \equiv u_{,j}(x) = \iiint_V \frac{\partial G(y,x)}{\partial x_j}\rho(y)dy \equiv u_{,j}(\rho)(x), j = 1,2,3. \tag{7}$$

We note that $u_{,j}$ are linear completely continuous operators from $L_p(V)$ в $C_\alpha(\underline{V})$ when $p > 3$. In order to find the second derivatives $u(x)$, we will use the formula (15) from (Novozhilov, 1948). As a result, a.e. in V we get a notion

$$u_{k,kj}(\rho_k)(x) = -\frac{1}{3}\delta_{kj}\rho_k(x) + \frac{1}{4\pi}\iiint_{E_3} \frac{f_{kj}\left(\frac{y-x}{|y-x|}\right)}{|y-x|^3}\rho_k^*(y)dy, j,k = 1,2,3, \tag{8}$$

$$f_{kj}\left(\frac{y-x}{|y-x|}\right) = \frac{3(y_k - x_k)(y_j - x_j) - \delta_{kj}|y-x|^2}{|y-x|^2},$$

where $\rho_k^*(y) = \rho_k(y)$ when $y \in V$, $\rho_k^*(y) = -\left(\frac{R^5}{|y|^5}\right)\rho_k\left(\frac{R^2}{|y|^2}y\right)$ when $y \in V_1$, $\rho_k^*(y) = -\rho_k\left(\underline{y}\right)$ when $y \in V_2$, $\rho_k^*(y) = \left(\frac{R^5}{|y|^5}\right)\rho_k\left(\frac{R^2}{|y|^2}\underline{y}\right)$ when $y \in V_3$; E_3 - three-dimensional Euclidean space; $\delta_{kj} = 1$ when $k = j$ и $\delta_{kj} = 0$ when $k \neq j$.

It should be noted that the function $f_{kj}\left(\frac{y-x}{|y-x|}\right)$ is a characteristic of a singular operator $u_{k,kj}$ (Timergalyev et al 2014). Having designated $\theta = (y-x)/|y-x| = (\theta_1, \theta_2, \theta_3), \theta_j = (y_j - x_j)/|y-x|$, $j = 1,2,3$, the characteristic can be expressed as $f_{kj}(\theta) = 3\theta_k\theta_j - \delta_{kj}, k, j = 1,2,3$. Direct calculations show that $\iint_{S_1} f_{kj}(\theta)ds = 0$; furthermore, it is evident that $\iint_{S_1}\left|f_{kj}(\theta)\right|^q ds \leq const,\ k, j = 1,2,3,\ 1/p + 1/q = 1,\ p > 3, S_1$ - a singular sphere. Therefore (Timergalyev et al 2014), $u_{k,kj}$ the essence of bounded operators is in $L_p(V), p > 3$.

Relations (6), (7), (8) are introduced in (3). As a result, in order to determine the function $\rho = (\rho_1, \rho_2, \rho_3)$ we arrive at a system of three-dimensional nonlinear singular integral equations of form

$$\rho_k(x) - \frac{\beta(x)}{4\pi}\iiint_{E_3}\frac{f_{kj}(\theta)}{|y-x|^3}\rho_j^*(y)dy - l_k(\rho) = g_k(\rho) + F_k(x), x \in V, \tag{9}$$

$$F_k(x) = \frac{3(1-2\nu)(1+\nu)}{(2-3\nu)E}X_k(x), l_k(\rho) \equiv l_k(u(\rho)), g_k(\rho) \equiv g_k(u(\rho)),$$

$$\beta(x) = \frac{3}{4-6\nu}, \qquad k = 1,2,3.$$

Based on relations (4) and given above set operator properties $u_{,j}$, $u_{k,kj}$, $j,k = 1,2,3$, and conditions $a), b)$, we easily establish that l_k - inear completely continuous, g_k - non-linear bounded operators in $L_p(V)$; $F_k(x) \in L_p(V), p > 3, k = 1,2,3$.

We will follow (Krasnoselsky, 1956) when studying the solvability of a system (9), in which the right-hand side is temporarily considered fixed. The study of the solvability of multidimensional singular integral equations is based on the calculation of the symbol of singular operators. We will define a singular operator symbol with $\Phi_{kj}(x,\theta)$.

$$A_{kj}\rho_j = \delta_{kj}\rho_j - \frac{\beta(x)}{4\pi}\iiint_{E_3}\frac{f_{kj}(\theta)}{|y-x|^3}\rho_j^*(y)dy - \delta_{kj}l_j(\rho), x \in V, j,k = 1,2,3$$

(there is no summation with j).

We will calculate $\Phi_{kj}(\theta)$. We will be using a formula (Timergalyev et al 2014):

$$\Phi_{kj}(x,\theta) = \delta_{kj} - \frac{\beta(x)}{4\pi}\iiint_{E_3}\frac{f_{kj}(y/|y|)}{|y|^3}e^{-i(y,z)}dy, \tag{10}$$

where $\theta = z/|z|, z = (z_1, z_2, z_3), (y,z) = y_1z_1 + y_2z_2 + y_3z_3 = |y||z| \cos\cos\gamma$ - scalar product of vectors y, z; γ - an angle between y, z; i - an imaginary unit.

By calculating the integrals in (10), we derive $\Phi_{kj}(\theta)$ the relations for symbols $\Phi_{kj}(\theta)$

$\Phi_{jj}(x,\theta) = \beta(1 - 2\nu + \theta_j^2), \Phi_{jk}(x,\theta) = \beta\theta_j\theta_k, j \neq k, \theta_j = \frac{z_j}{|z|}, j,k = 1,2,3.$

Based on the theorem 3.40 from (Novozhilov, 1948), we derive

$$\Delta_1 = \Phi_{11}(x,\theta) = \beta(1-2\nu+\theta_1^2),\ \Delta_2 = det(\Phi_{jk})_{2\times 2} =$$

$$= \beta^2[(1-2\nu)^2 + (1-2\nu)(\theta_1^2+\theta_2^2)],$$

$$\Delta_3 = det(\Phi_{jk})_{3\times 3} = 2\beta^3(1-2\nu)^2(1-\nu). \tag{11}$$

Let a Poisson's ratio $\nu = \nu(x)$ satisfy the condition

$$-1 < \nu(x) \le \nu_0 < 1/2 \ \ \forall x \in \underline{V},\ \nu_0 = const. \tag{12}$$

Based on (11), we easily obtain

$$|\Delta_1| > 0{,}3(1-2\nu_0), |\Delta_2| > [0{,}3(1-2\nu_0)]^2, |\Delta_3| > 2(1-\nu_0)(0{,}3)^3(1-2\nu_0)^2$$

$$\forall x \in \underline{V},\ \forall\theta \in S_1,$$

which suggests that the exact lower bounds of the determinants moduli Δj are positive. Therefore (Novozhilov, 1948), the system index (9) is zero with the Fredholm alternative applicable to it. Let $\rho = (\rho_1,\rho_2,\rho_3) \in L_p(V), p > 3$ - A non-trivial solution of the system (9) with a zero right-hand side: $g_k(\rho) + F_k(x) \equiv 0, k = 1,2,3$. This solution by the formula (6) corresponds to the displacement vector $u = (u_1,u_2,u_3) \in W_p^{(2)}(V), p > 3$ that satisfies the boundary condition (5) and a.e. a system of homogeneous linear equations

$$\sigma_{e,j}^{kj} = 0,\ k = 1,2,3, \tag{13}$$

where $\sigma_e^{jj} = 2\mu e_{jj} + \lambda(e_{11}+e_{22}+e_{33}), \sigma_e^{jk} = \mu e_{jk}, j \neq k, j,k = 1,2,3.$

Equations in (13) are respectively multiplied by u1, u2, u3, integrated by V and summed. Then, taking into account (5), we do integration by parts. Finally, we obtain

$$\iiint_V \{(1-2\nu)[(\sigma_e^{11})^2 + (\sigma_e^{22})^2 + (\sigma_e^{33})^2] + 2(1+\nu)[(\sigma_e^{12})^2 + (\sigma_e^{22})^2 + (\sigma_e^{13})^2]\}dV = 0,$$

which suggests $e_{jk} = 0, k = 1,2,3,$ and, therefore, $u_k = 0, k = 1,2,3$. So $\rho = 0$ a.e. in V.

Thus, there is an inverse operator $(I-P)^{-1}$ bounded in $L_p(V), p > 3$, with the help of which (9) reduces to an equivalent system of the form

$$\rho - G\rho = 0, \tag{14}$$

where notations are accepted: $G\rho = (I-P)^{-1}(g(\rho)+F)$, $P\rho = (P_1\rho, P_2\rho, P_3\rho)$, $F = (F_1,F_2,F_3)$, $g(\rho) = (g_1(\rho), g_2(\rho), g_3(\rho))$,

$$P_k\rho = \frac{\beta}{4\pi}\iiint_{E_3} \frac{f_{kj}(\theta)}{|y-x|^3}\rho_j^*(y)dy + l_k(\rho), \theta = \frac{y-x}{|y-x|}, k = 1,2,3.$$

Exists

Lemma. Let the conditions $(a),(b)$, inequality (12) be satisfied. Then G is a nonlinear bounded operator in $L_p(V), p > 3$, besides, for any $\rho^j (j = 1,2) \in L_p(V), p > 3$ belonging to the ball $\|\rho^j\|_{L_p(V)} < r$, the following evaluation is fair $\|G(\rho^1) - G(\rho^2)\|_{L_p(V)} \le (q_1+q_2 r)r\|\rho^1-\rho^2\|_{L_p(V)}$, where q_j $(j = 1,2)$ - known constants that are not dependent on r.

Let us assume that the ball radius and the external forces acting on the elastic body are such that the conditions are satisfied

$$q = (q_1+q_2 r)r < 1, \|G(0)\|_{L_p(V)} < (1-q)r, G(0) = (I-P)^{-1}F. \tag{15}$$

Under these conditions (14) we can apply a contraction mapping principle [7, p.146], according to which the equation (14) in the ball $\|\rho\|_{L_p(V)} < r$ has the only possible solution $\rho \in L_p(V), p > 3$.

Knowing that $\rho = (\rho_1,\rho_2,\rho_3)$, from the formula (6), we find the solution $u = (u_1,u_2,u_3) \in W_p^{(2)}(V), p > 3$ of Problem A.

Thus, the following theorem has been proved.

The theorem. Let conditions a, (b) of inequality (12), (15) be satisfied. Then the nonlinear boundary-value problem for an elastic isotropic inhomogeneous hemisphere under kinematic boundary conditions has a unique generalized solution in some ball of space $W_p^{(2)}(V), p > 3$.

4 Summary

The solvability of spatial boundary value problems of the elasticity theory is very relevant and is being carried out in two main directions. The first direction is characterized by the use of functional analysis methods (the Hilbert space method, variational methods, implicit function theorems), which allow us to study the existence of generalized solutions to a wide range of problems in the theory of elasticity in various energy spaces. The second direction is based on the theory of singular integral equations, which is based on fundamental solutions of equilibrium equations. Currently, such fundamental solutions are constructed for equations with constant and piecewise-constant coefficients that describe the equilibrium state of isotropic homogeneous and piecewise-homogeneous elastic bodies. The research proposal of this academic paper which concerns three-dimensional problems is developed in the second direction (Pakdel & Talebbeydokhti, 2018: Deyhim & Zeraatkish, 2016).

5 Conclusions

We have proved the existence theorem and have developed the analytical method for finding solutions of geometrically nonlinear spatial boundary value problems for an elastic isotropic inhomogeneous hemisphere under kinematic boundary conditions.

Acknowledgements

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.

Literature

1. Vorovich I.I.: Mathematical problems of the non-linear shallow-shell theory. *Nauka*, Moscow, 1989.
2. Mikhlin S.G. Multidimensional singular integrals and integral equations. L.: Physmathgiz, 1962, 256 P.
3. Novozhilov V.V.: Fundamentals of the non-linear elasticity theory. L. - M.: Gostechizdat, 1948, 576 P.
4. Timergalyev S.N., Uglov A.N., Yakupova G.A. A study of solvability of spatial boundary value problems for a ball in the linear elasticity theory. Scientific and technical vestnik of Volga region, No.1, 2014. 41-44. p

5. Timergalyev S.N., Yakupova G.A. The existence of solutions of spatial non-linear boundary value problems for an elastic inhomogeneous ball Grid methods for boundary value problems and applications. Materials of X International Conference (Kazan, September, 2014), Kazan: The publishing house of Kazan State University, 2014, 586- 591.p.
6. *Yakupova Gulnara A.:* Solvability of a Spatial Nonlinear Boundary Value Problem for Elastic Inhomogeneous Isotropic Ellipsoid, HELIX- Vol.8, Is.6. 2018. 4688- 4691.p.
7. Krasnoselsky M.A. Topolygic methods in the theory of nonlinear integral equations. M.: Gostechizdat, 1956, 392 P.
8. Ilyasov Y.Sh., Valeev N.F. On nonlinear boundary value problem corresponding to N-dimensional inverse spectral problem. Journal of Differential Equations
Volume 266, Issue 8, 5 April 2019, 4533-4543.p.
9. Polyanin, A D.: Inna K. Shingarevad. Hypersingular nonlinear boundary-value problems with a small parameter. Applied Mathematics Letters. Volume 81, July 2018, 93-98. P.
10. Haynes, RD.: Faysol Ahmed. Linearized domain decomposition approaches for nonlinear boundary value problems. Journal of Computational and Applied Mathematics. Volume 346, 15 January 2019, 620-637.p.
10. Pakdel, M., Talebbeydokhti, A.: The Effect of Adjustment Announcement of Predicted Profit on Price and Trading Volume of Listed Companies in Tehran Stock Exchange. Dutch Journal of Finance and Management, *2*(1), 2018. 49. https://doi.org/10.29333/djfm/5825.
11. Deyhim, T., Zeraatkish, y.: Investigate the trend of rural development in Gachsaran city with Morris method. UCT Journal of Management and Accounting Studies, 4(1), 2016. 23-28 p.

Primary Paper Section: B

Secondary Paper Section: BA, BB, BC