ANALYSIS OF THE SOLVABILITY OF A SPATIAL NONLINEAR BOUNDARY VALUE PROBLEM FOR AN ARBITRARY ELASTIC INHOMOGENEOUS ISOTROPIC BODY

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Abstract: The work is devoted to the proof of the existence theorem and the development of analytical, numerical methods for finding solutions to geometrically nonlinear boundary value problems of the three-dimensional theory of elasticity. At present, the solvability of nonlinear spatial boundary-value problems for isotropic homogeneous and piecewise-homogeneous elastic bodies is most fully studied. Therefore, it is very urgent to develop mathematical methods that allow us to study solvability and prove existence theorems for solutions of spatial nonlinear problems for anisotropic inhomogeneous elastic bodies. In this paper, we study the solvability of nonlinear boundary value problems of the three-dimensional theory of elasticity for an isotropic inhomogeneous arbitrary body under kinematic boundary conditions. The basis of the proposed study in the case of three-dimensional problems is based on integral representations for displacements based on the fundamental Laplace solutions, with the help of which the equilibrium equations are reduced to a system of three-dimensional singular integral equations with respect to the volume occupied by the elastic body. The solvability of the system of integral equations is established using topological methods. The representations constructed in this way for displacements make it possible to reduce the initial system of equilibrium equations to one nonlinear operator equation, the solvability of which is studied using the principle of compressed mappings.

Keywords: Elastic inhomogeneous isotropic body, equilibrium equations, boundary value problem, three-dimensional singular integral equations, symbol of a singular operator, existence theorem.

1 Introduction

When developing computer programs for solving complex problems of calculating elastic structures, it is always necessary to select a real model for existing processes. The solution to this problem is based on a rigorous mathematical study of the solvability of boundary value problems. The existence of existence theorems makes it easy to prove the convergence of numerical methods to an exact real solution (Novozhilov, 1948). Based on this, a rigorous study of the solvability of boundary value problems and the proof of existence theorems are a very urgent problem in the mathematical theory of elasticity (Timergaliev et al, 2014; Timergaliev, 2014; Timergaliev & Yakupova, 2014). In this paper, to study the solvability of nonlinear boundary value problems for an isotropic inhomogeneous arbitrary body, we use a method based on the use of integral representations for displacement components (Ilyasov & Valeev, 2019). The problem reduces to a system of singular integral equations over a ball, the solvability of which is established by involving the symbol of the singular operator (Yakupova, 2018).

2 Methods

To study the solvability of spatial nonlinear boundary value problems, a method is proposed that is based on integral representations for displacements. An approach based on the use of the harmonic Green function of the Dirichlet problem in the case of elastic bodies of a special configuration (ball, half-space, cylinder, etc.) and the theory of harmonic potential in the case of arbitrary elastic bodies are presented. A distinctive feature of the proposed method is that the fundamental solutions underlying the theory of potential are not related to the original system of equilibrium equations, they are only solutions of the Laplace equation (Polyanin & Shingarevad, 2019). The integral representations thus obtained determine the displacements satisfying the given boundary conditions and the Poisson equation with an arbitrarily fixed right-hand side. The equilibrium equations are fulfilled by choosing the right-hand side of the Poisson equation, to determine which a system of nonlinear three-dimensional singular integral equations is derived for the volume occupied by the elastic body, which is equivalent to the original system of equilibrium equations (Haynes & Ahmed, 2019). To study the solvability of the system of integral equations, the theory of multidimensional integral equations developed by prof. S.G. Mikhlin (Mikhlin, 1962).

3 Results and Discussion

In an arbitrary simply connected bounded region V occupied by an elastic body, we consider a system of nonlinear differential equations of the form:

(1)

1.(hereinafter, for repeating Latin indices, a summation from 1 to 3 is carried out), in which the notation is accepted:

$f_1 = \frac{\partial}{\partial x_j}(\sigma^{j3}\omega_2 - \sigma^{j2}\omega_3),\ f_2 = \frac{\partial}{\partial x_j}(\sigma^{j1}\omega_3 - \sigma^{j3}\omega_1), f_3 = \frac{\partial}{\partial x_j}(\sigma^{j2}\omega_1 - \sigma^{j1}\omega_2);$

(2)

$\varepsilon_{kj} = e_{kj} + æ_{kj}, e_{jj} = u_{j,j}, e_{jk} = u_{j,k} + u_{k,j}, æ_{jj} = (\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_j^2)/2,$

$æ_{kj} = -\omega_k\omega_j, j \neq k, j, k = \overline{1,3}; \omega_1 = (u_{3,2} - u_{2,3})/2,$

$\omega_2 = (u_{1,3} - u_{3,1})/2,\ \omega_3 = (u_{2,1} - u_{1,2})/2;$ $\mu = \frac{E}{2(1+\nu)},$ $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},$

the symbol $\sigma^{kj}_{,j}$ in (1) means the partial derivative $\sigma^{kj}_{,j} \equiv \partial\sigma^{kj}/\partial x_j$.

The system of equations (1) together with relations (2) describes the equilibrium state of an elastic isotropic inhomogeneous arbitrary body. At the same time: $\sigma^{kj} = \sigma^{jk}$ — stress components, $\varepsilon_{kj} = \varepsilon_{jk}$ — strain components, ω_k — element rotation angles around the axis Ox_k, $u = (u_1, u_2, u_3)$ — displacement vector, $u_{j,k} \equiv \partial u_j/\partial x_k$, $j, k = \overline{1,3}$; $X_k (k = \overline{1,3})$ — components of volumetric external forces acting on the elastic body; μ - shear modulus, λ - Lame parameter, $E = E(x)$ - tensile modulus, $\nu = \nu(x)$ - Poisson's ratio, $x = (x_1, x_2, x_3)$ - rectangular Cartesian coordinates of the body point in the region V.

If in system (1) stresses and strains are replaced by expressions from (2), then we obtain a system of equations of equilibrium in displacements:

(3)

Where

$$l_k(u) = [\mu_{,k}e_{kk} + \mu_{,j}e_{kj} + \lambda_{,k}(e_{11} + e_{22} + e_{33})]/\mu,$$

$$g_k(u) = \frac{1}{\mu}\Big\{f_k(u) + \frac{\partial}{\partial x_k}[(\mu+\lambda)(\chi_{11} + \chi_{22} + \chi_{33})] +$$

∂∂xjμχjk, θ=div u;

Δ - Laplace operator.

Task I. It is required to find a solution $u = (u_1, u_2, u_3)$ of the system (1) in the field V, satisfying condition∂Von its border

(4)

We will study Problem I in a generalized setting. Let the following conditions be satisfied: (a) $E(x), \nu(x) \in W_p^{(1)}(V), p > 3$; (b) $X_k \in L_p(V), p > 3, k = \overline{1,3}$.

Definition. A generalized solution to Problem I is the displacement vector $u=(u_1,u_2,u_3)\in W_p^{(2)}(V)$, $p>3$, almost everywhere (a.e.) satisfying system (1) and the boundary condition (4).

Here $W_p^{(j)}(V)(j=1,2)$ are Sobolev spaces. By virtue of embedding theorems for Sobolev spaces $W_p^{(j)}(V)$ with $p>3$ generalized solution $u\in C_\alpha^1(\bar{V})$, and elastic characteristics $B^{jknm}\in C_\alpha(\bar{V}), \alpha=(p-3)/p, p>3$.

Let $y=\varphi(x)=(\varphi_1(x),\varphi_2(x),\varphi_3(x))(y=(y_1,y_2,y_3))$ be a one-to-one mapping of the region V onto the ball $S_R: y_1^2+y_2^2+y_3^2\le R^2$. By $x=\psi(y)=(\psi_1(y),\psi_2(y),\psi_3(y))$ we denote the map opposite to $y=\varphi(x)$. We assume that

$$\varphi(x)\in W_p^{(2)}(V),\ \psi(y)\in W_p^{(2)}(S_R),\ p>3\ . \quad (5)$$

In (3), we pass to the new variables $y_j, j=\overline{1,3}$. Then in ball S_R we obtain a system of equations of the form

(6)

where

$$l_k(u)=\mu\left[u_{ky_n}1^j\varphi_{nx_jx_j}+u_{jy_n}\varphi_{nx_kx_j}\right]+e_{kk}(x)\mu_{y_n}\varphi_{nx_k}+e_{kj}(x)\mu_{y_n}\varphi_{nx_j}+1^je_{jj}(x)\lambda_{y_n}\varphi_{nx_k},$$

$$g_k(u)=f_k(u)+(\mu_{y_n}+\lambda_{y_n})\varphi_{nx_k}1^j\chi_{jjy_n}(x)\varphi_{nx_k}+\mu_{y_n}\varphi_{nx_j}\chi_{jky_n}(x)\varphi_{nx_j}, x=\psi(y),$$

$$a_{kj}^{nm}\equiv a_{kj}^{nm}(y)=b_{kj}^{qs}(x)\varphi_{nx_q}(x)\varphi_{mx_s}(x),$$

(7)

symbol $1^j\varphi_{nx_jx_j}$ means summation by j: $1^j\varphi_{nx_jx_j}=\sum_{j=1}^3\varphi_{nx_jx_j}$.

Note that $l_k(u), g_k(u)$ are respectively linear completely continuous and nonlinear bounded operators in $L_p(S_R), p>3$.

A solution to system (6) in the ball $S_R$satisfying condition (4) on its boundary $\partial S_R: y_1^2+y_2^2+y_3^2=R^2$ will be sought in the form

(8)

where $\rho=(\rho_1,\rho_2,\rho_3)$ is an arbitrary vector function belonging to the space $L_p(S_R), p>3$; $G(x,y)$ is the harmonic Green function of the Dirichlet problem for the ball S_R.

We substitute relation (8) into (6) and to determine the function $\rho=(\rho_1,\rho_2,\rho_3)$ we arrive at a system of three-dimensional nonlinear singular integral equations of the form

(9)

Where

$P_{kj}(\rho_j)=-\frac{1}{3}(a_{kj}^{11}+a_{kj}^{22}+a_{kj}^{33})\rho_j(x)+\frac{1}{4\pi}a_{kj}^{nm}\iiint_{E_3}\frac{f_{nm}(\theta)}{|x-y|^3}\rho_j^*(x)\,dx+l_k(\rho), f_{nm}(\theta)=3\theta_n\theta_m-\delta_{nm}, \theta_j=(x_j-y_j)/|x-y|, \theta=(\theta_1,\theta_2,\theta_3), \delta_{nm}=1$ при $n=m$ и $\delta_{nm}=0$ при $n\ne m; n,m,k,j=\overline{1,3}$; $\rho_j^*(x)=\rho_j(x)$ at $x\in S_R$ and $\rho_j^*(x)=-(R^5/|x|^5)\rho_j(R^2/|x|^2)$ at $x\,\bar{\in}\,\overline{S_R}$, E_3 - three-dimensional Euclidean space.

Note that $P_{kj}(\rho_j)$ are linear bounded operators in $L_p(S_R), p>3$.

In studying the solvability of system (9) we will follow (Mikhlin, 1962) By $\Phi_{kj}(y,\theta)$ we denote the symbol of the singular operator $P_{kj}(\rho_j)$. It can be shown that the symbol $\Phi_{kj}(y,\theta)$ is given by the formula

(10)

where

$\gamma_n=\varphi_{1x_n}(x)\theta_1+\varphi_{2x_n}(x)\theta_2+\varphi_{3x_n}(x)\theta_3,\ n=\overline{1,3}, x=\psi(y), y\in S_R, x\in V.$

Introducing (7) into (10), we have

$$\Phi_{11}(y,\theta)=-\alpha\gamma_1^2-\mu\gamma_2^2-\mu\gamma_3^2, \Phi_{22}(y,\theta)=-\mu\gamma_1^2-\alpha\gamma_2^2-\mu\gamma_3^2,$$

$$\Phi_{33}(y,\theta)=-\mu\gamma_1^2-\mu\gamma_2^2-\alpha\gamma_3^2, \Phi_{jk}(y,\theta)=-2\beta\gamma_j\gamma_k, j\ne k, j,k=\overline{1,3}\ .$$

We introduce the determinants $\Delta_1=\Phi_{11}(y,\theta)$, $\Delta_2=det\left(\Phi_{kj}(y,\theta)\right)_{2\times2}$, $\Delta_3=det\left(\Phi_{kj}(y,\theta)\right)_{3\times3}$. For them we get the following expressions:

$\Delta_2=\gamma_1^2[\mu\alpha\gamma_1^2+\mu_0\gamma_2^2+\mu(\mu+\alpha)\gamma_3^2/2]+\gamma_2^2[\mu_0\gamma_1^2+\mu\alpha\gamma_2^2+$ *μμ+αγ32/2+*

$$+\gamma_3^2[\mu(\mu+\alpha)(\gamma_1^2+\gamma_2^2)/2+\mu^2\gamma_3^2],$$

$$\Delta_3=-\{\gamma_1^4[\mu^2\alpha\gamma_1^2+\alpha_0(\gamma_2^2+\gamma_3^2)]+\gamma_1^2\gamma_2^2[\alpha_0(\gamma_1^2+\gamma_2^2)+\beta_0\gamma_3^2]+$$

(11)

$+\gamma_2^2\gamma_3^2[\alpha_0(\gamma_2^2+\gamma_3^2)+\beta_0\gamma_1^2]+\gamma_3^4[\mu^2\alpha\gamma_3^2+\alpha_0(\gamma_1^2+\gamma_2^2)]\}$,

where

$$\mu_0=(\mu^2+\alpha^2-4\beta^2)/2, \alpha_0=\mu(\mu\alpha+\mu_0)/2, \beta_0=2(\mu^3+8\beta^3-\alpha^3+3\alpha\mu_0)/3.$$

Let the Poisson's ratio $\nu=\nu(x)$ satisfy the condition

(12)

Then, under the conditions (a) and (12), the functions $\alpha=\alpha(x)$, $\mu=\mu(x)\ge c>0\ \forall x\in\bar{V}$. In addition,, , by direct calculation, we see that $\mu_0=\mu_0(x)$, $\alpha_0=\alpha_0(x)$, $\beta_0=\beta_0(x)\ge c>0\ \forall x\in\bar{V}$. Then the expressions in square brackets in (11) are positive definite quadratic forms with respect to the variables $\gamma_1,\gamma_2,\gamma_3$. Therefore $|\Delta_j|\ge c>0\ (j=\overline{1,3})\ \forall y\in\overline{S_R}, \forall\theta\in\partial S_1$. Therefore, the exact lower bounds of the modules of the determinants Δ_j $(j=\overline{1,3})$ are positive. Then the index of the system of equations (9) is equal to zero and the Fredholm alternative is applicable to it (Mikhlin, 1962). As a result, system (9) is reduced to an equivalent system of the form

(13)

where $G\rho$ is the nonlinear bounded operator in $L_p(S_R), p>3$, and, for any $\rho^j(j=1,2)\in L_p(S_R), p>3$, belonging to the ball $\|\rho^j\|_{L_p(S_R)}<r$, the estimate $\|G(\rho^1)-G(\rho^2)\|_{L_p(S_R)}\le(q_1+q_2r)r\|\rho^1-\rho^2\|_{L_p(S_R)}$is valid, where q_j $(j=1,2)$ are known constants independent of r.

Suppose that the radius r of the ball and the external forces acting on the elastic body are such that the conditions

(14)

Under these conditions, the principle of squeezed mappings can be applied to equation (13) (Krasnoselsky, 1956), according to which equation (13) in the ball $\|\rho\|_{L_p(S_R)} < r$ has a unique solution $\rho \in L_p(S_R), p > 3$. Knowing $\rho = (\rho_1, \rho_2, \rho_3)$, we find the solution $u = (u_1, u_2, u_3) \in W_p^{(2)}(S_R), p > 3$ of problem I using formula (8).

Thus, the following main theorem is proved.

Theorem. Let conditions $(a), (б)$ *and (5) be satisfied, inequality (14). Then problem I for an elastic inhomogeneous isotropic ellipsoid has a unique generalized solution* $u = (u_1, u_2, u_3) \in W_p^{(2)}(V), p > 3$.

4 Summary

Studies of the solvability of spatial boundary value problems of the theory of elasticity are carried out in two main directions. The first direction is based on the application of functional analysis methods (the Hilbert space method, variational methods, implicit function theorems), which allow us to study the existence of generalized solutions to a wide class of problems in the theory of elasticity in various energy spaces. On this path, the existence theorems of solutions of linear boundary value problems (G. Fiker. Existence theorems in the theory of elasticity.-M.: Mir, 1974.-160 pages; G. Duvo, J.-L. Lyons. Inequalities in mechanics and physics, -M.: Nauka, 1980.-384 p.), As well as nonlinear boundary value problems of the theory of elasticity (Sjarle F. Mathematical theory of elasticity, Transl. From English.-M.: Mir, 1992.-472 with.). Studies of the second direction are based on the theory of singular integral equations (V.D. Kupradze, T.G. Hegelia, M.O. Basheleishvili, T.V. Bourguladze. Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity.-M .: Nauka, 1976.-664 pages.), Which are based on fundamental solutions of the equilibrium. equations. Currently, such fundamental solutions are constructed for equations with constant and piecewise-constant coefficients that describe the equilibrium state of isotropic homogeneous and piecewise-homogeneous elastic bodies. The study proposed in the framework of this work concerning three-dimensional problems is the development of research in the second direction.

5 Conclusions

An existence theorem is proved and an analytical method is developed for finding solutions of nonlinear boundary value problems of the three-dimensional theory of elasticity for an isotropic inhomogeneous arbitrary body under kinematic boundary conditions.

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