ABOUT STABILITY AND ACCURACY OF FUNCTIONING OF SYSTEMS WITH DISTRIBUTED AND CONCENTRATED PARAMETERS

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Abstract: Sufficient stability conditions are obtained in the form of inequalities connecting the system coefficients (parameters) and the condition for the accuracy of its functioning. Functioning accuracy is understood as the requirement that the deviations of some system state variables from their calculated values remain within predetermined limits. The results obtained in the article allow us to study the stability and accuracy of the functioning of various complex engineering objects with distributed and concentrated parameters. As an example, the stability of a rotor-type wind turbine with a load (generator, pump, etc.) and the elasticity of the shaft transmitting the torque from the wind turbine to the load are considered. The use of environmentally friendly wind turbines to reduce energy costs is a promising area.

Keywords: systems with distributed and concentrated parameters, stability, functioning accuracy, Lyapunov function method.

1 Introduction

One of the main methods for studying the system's stability with distributed parameters is the method of Lyapunov functions (functionals). At the same time, along with purely distributed systems, systems with distributed and concentrated parameters were also considered. A fairly complete review and problem status in this area can be found in (Sirazetdinov, 1987; Wang, 1968; Wang, 1966; Bayramov, 1995). Along with theoretical studies using the Lyapunov function method, studies of specific objects with distributed parameters are carried out. For example, elastic and aeroelastic objects (Bayramov, 1995; Parks, 1967; Wang, 1966; Meirovitch, 1970), chemical reactors liquid rocket engines (Sirazetdinov, 1987; Bayramov, 1995). In applications, the main difficulty is the construction of the corresponding Lyapunov functionals, which, when studying concrete objects, were usually constructed intuitively based on the total energy, the first integrals, and other considerations. In solving systems stability with distributed parameters problems, it was proposed in (Bayramov, 1995 ; Bairamov et al., 2016), that the initial equations in partial derivatives of high order be preliminarily reduced to a system of first-order equations concerning time and spatial coordinates. Further, for this system, Lyapunov functionals are constructed according to specific equations in the form of integral quadratic forms, which sign-definiteness can be checked using the well-known Sylvester criterion. Such an approach allows the constructive construction of Lyapunov functionals and significantly expands the possibilities of using the Lyapunov function method in specific applications.

Technical conditions for the system functioning along with stability often require that the one or more state variables deviations from their calculated values remain within specified limits (accuracy of operation). Moreover, deviations of other variables are not strictly controlled. For example, in hydraulic systems, it is important to control the fluid flow rate supplied to the consumer, in pneumatic systems - pressure.

In this paper, the idea of transforming the original high-order equations into a system of first-order equations is used to study the stability and the systems functioning accuracy with distributed and concentrated parameters.

2 Methods

The work uses universally recognized mathematically rigorous and accurate research methods. The main ones are methods for converting high-order differential equations into a system of first-order equations and the Lyapunov function method.

When calculating the derivative of the Lyapunov function (2.1) by equations (1.1) – (1.3), the modified Lagrange multiplier method is used to take into account equations without derivatives.

The conditions of stability and accuracy of operation are written based on well-known classical results from the theory of stability of finite-dimensional and distributed systems. The sign-definiteness check of ordinary and integral quadratic forms is carried out according to the Sylvester criterion.

3 Results And Discussion

1. Statement of the problem. We consider a system with one distributed and other finite-dimensional links, the disturbed state of which is described by the equations

$$\frac{\partial\varphi}{\partial t}=A(x)\frac{\partial\varphi}{\partial x}+B(x)\frac{\partial\psi}{\partial x}+A_0(x)\varphi+B_0(x)\psi,$$

$$C(x)\frac{\partial\varphi}{\partial x}+D(x)\frac{\partial\psi}{\partial x}+C_0(x)\varphi+D_0(x)\psi=0,$$

$$x\in(0,1), \tag{1.1}$$

$$\frac{dz}{dt}=F_1z+F_2\varphi(0,t)+F_3\varphi(1,t), \tag{1.2}$$

$$\Gamma_1\varphi(0,t)=\Gamma_2z,\quad \Gamma_3\varphi(1,t)=\Gamma_4z,$$

$$t\in I=(0,\infty), \tag{1.3}$$

where $\varphi=\varphi(x,t)$ – n is a dimensional vector of state variables of a distributed link, $\psi=\psi(x,t)$ – l is a dimensional vector of state variables of this link, the time derivatives of which are not included in system (1.1), $z=z(t)$ – m is a dimensional vector of state variables of finite-dimensional links, $A(x)$, $B(x)$, $C(x)$, $D(x)$, $A_0(x)$, $B_0(x)$, $C_0(x)$, $D_0(x)$ are matrices whose elements are limited continuous functions, F_1, F_2, F_3, Γ_1, Γ_2, Γ_3, Γ_4 are constant matrices.

From a mathematical point of view, problem (1.1) – (1.3) is a boundary value problem for partial differential equations. Equations (1.1) are the general form for writing any linear partial differential equation of arbitrary order in the form of a system of partial differential equations of the first order [4, 8]. To convert high-order equations to the form (1.1), we must take the lower derivatives as additional variables and write the initial equation and integrability condition in these variables.

Equations (1.3) are simple boundary conditions connecting the boundary values of the $\varphi(x,t)$ components with each other or with the variable z. The dynamic equation (1.2) of finite-dimensional units located at both ends of the distributed unit contains the boundary values $\varphi(x,t)$ and represents a complex boundary condition in the form of a differential equation.

Equations of the type (1.1) – (1.3) describe systems having elastic shafts of considerable length, for example, between the engine and the working machine (generator, pump, compressor, etc.); systems containing pipelines (highways) in which it is necessary to take into account the flow of liquid or gas distributed nature, etc.

We introduce the metric

$$\rho = \int_0^1 \varphi^T \varphi \, dx \tag{1.4}$$

characterizing the distributed link perturbed state, and consider the system stability and accuracy functioning problem (1.1) – (1.3).

The task. It is required to find conditions under which system (1.1) – (1.3) is asymptotically stable concerning the variables ρ, z, and any its solution with initial data from the domain

$$\rho(t_0) < H_{00}, \quad |z_i(0)| < H_{0i}, \quad i = \overline{1,m} \tag{1.5}$$

satisfies the condition

$$|z_1(t)| < H_1, \quad t > 0, \tag{1.6}$$

where $H_{0i}\left(i = \overline{0,m}\right), \quad H_1(H_1 > H_{01})$ are the given positive numbers.

Here, for definiteness, the deviations of only one variable z_1 are controled.

1. The solution to the problem. To solve the problem we use the Lyapunov function

$$V = V_1 + V_2 = \int_0^1 \varphi^T(x,t) v(x) \varphi(x,t) dx + z^T(t) Q z(t), \tag{2.1}$$

where $v(x)$, Q are symmetric matrices: the elements Q are constant, and the elements $v(x)$ are continuously differentiable bounded functions.

The second equation (1.1) and equations (1.3) do not contain time t derivatives. This does not allow directly to calculate the derivative V due to the whole system. First, we calculate the derivative dV/dt by the first equation (1.1) and equation (1.2):

$$\frac{dV}{dt} = \int_0^1 \Big[\varphi^T v \left(A\frac{\partial \varphi}{\partial x} + B\frac{\partial \psi}{\partial x} \right) + \left(\frac{\partial \varphi^T}{\partial x} A^T + \frac{\partial \psi^T}{\partial x} B^T \right) v\varphi + \varphi^T \left(vA_0 + A_0^T v \right)\varphi +$$
$$+ \varphi^T v B_0 \psi + \psi^T B_0^T v \varphi \Big] dx + z^T \left(QF_1 + F_1^T Q \right) z + 2\varphi^T(0,t) F_2^T Q z +$$
$$+ 2\varphi^T(1,t) F_3^T Q z + 2\psi^T(0,t) F_4^T Q z + 2\psi^T(1,t) F_5^T Q z. \tag{1.7}$$

Using the modified Lagrange multiplier method, to take into account the second equation (1.1) and equations (1.3), we add to this derivative

$$\int_0^1 \Big[\left(\varphi^T P_1 + \psi^T P_2 \right) \left(C\frac{\partial \varphi}{\partial x} + D\frac{\partial \psi}{\partial x} + C_0 \varphi + D_0 \psi \right) +$$
$$+ \left(\frac{\partial \varphi^T}{\partial x} C^T + \frac{\partial \psi^T}{\partial x} D^T + \varphi^T C_0^T + \psi^T D_0^T \right) \left(P_1^T \varphi + P_2^T \psi \right) \Big] dx = 0, \tag{1.8}$$

$$\left[\varphi^T(0,t) R_1 + z^T R_2 \right] \left[\Gamma_1 \varphi(0,t) - \Gamma_2 z \right] + \left[\varphi^T(0,t) \Gamma_1^T - z^T \Gamma_2^T \right] \times \times \left[R_1^T \varphi(0,t) + R_2^T z \right] = 0,$$
$$\left[\varphi^T(1,t) R_3 + z^T R_4 \right] \left[\Gamma_3 \varphi(1,t) - \Gamma_4 z \right] + \left[\varphi^T(1,t) \Gamma_3^T - z^T \Gamma_4^T \right] \times \times \left[R_3^T \varphi(1,t) + R_4^T z \right] = 0, \tag{1.9}$$

where $P_1 = P_1(x)$, $P_2 = P_2(x)$, R_1, R_2, R_3, R_4 are while arbitrary matrices: P_1, P_2 are continuous, R_1, R_2, R_3, R_4 are constant. The brackets $\left(\varphi^T P_1 + \psi^T P_2\right)$, $\left(P_1^T \varphi + P_2^T \psi\right)$, $\left[\varphi^T(0,t) R_1 + z^T R_2\right]$, $\left[R_1^T \varphi(0,t) + R_2^T z\right]$ play the role of Lagrange multipliers.

We perform integration by parts and require that the matrices v, P_1, P_2, Q, R_1, R_2, R_3, R_4 satisfy the equations

$$vA + P_1 C = A^T v + C^T P_1^T, \quad P_2 D = D^T P_2^T, \tag{2.2}$$

$$vB + P_1 D = C^T P_2^T, \quad P_2 D_0 + D_0^T P_2^T = \frac{dP_2 D}{dx},$$

$$vB_0 + P_1 D_0 + C_0^T P_2^T = \frac{d\left(vB + P_1 D\right)}{dx}, \quad x \in (0,1). \tag{2.3}$$

and boundary conditions at $x = 0$ and $x = 1$:

$$v(0)A(0) + P_1(0)C(0) - R_1\Gamma_1 - \Gamma_1^T R_1^T = 0,$$

$$QF_2 + R_2\Gamma_1 - \Gamma_2^T R_1^T = 0,$$

$$v(1)A(1) + P_1(1)C(1) + R_3\Gamma_3 + \Gamma_3^T R_3^T = 0,$$

$$QF_3 + R_4\Gamma_3 - \Gamma_4^T R_3^T = 0,$$

$$(P_2 D)\Big|_0^1 = (vB + P_1 D)\Big|_0^1 = 0. \tag{2.4}$$

Then, for the derivative dV/dt, by system (1.1) – (1.3), we obtain the expression

$$\frac{dV}{dt} = -\int_0^1 \varphi^T w \varphi \, dx - z^T H z, \tag{2.5}$$

those a quadratic form of the same form as for V (2.1). Here

$$w = \frac{d\left(vA + P_1 C\right)}{dx} - vA_0 - A_0^T v - P_1 C_0 - C_0^T P_1^T, \tag{2.6}$$

$$H = -\left(QF_1 + F_1^T Q\right) + R_2\Gamma_2 + \Gamma_2^T R_2^T + R_4\Gamma_4 + \Gamma_4^T R_4^T. \tag{2.7}$$

The results obtained allow us to solve the problem of constructing the quadratic form V (2.1). To do this, one should set the symmetric matrix $w(x)$ and solve equations (2.2), (2.5) concerning the matrices v, P_1, P_2 under the boundary conditions arising from equations (2.3). However, unlike the problem of constructing quadratic forms in the case of linear ordinary differential equations, here not always all elements of the matrix w can be given arbitrarily, some of them are determined in the course of solving the problem. From equations (2.3) we also find the matrices Q, R_1, R_2, R_3, R_4.

According to the method of Lyapunov functions, the solution to the problem will be the conditions:

a) the integral quadratic form V_1 (2.1) is continuous and definitely positive in the metric ρ;

b) the usual quadratic form V_2 (2.1) is definitely positive;

c) the derivative (2.4) is definitely negative concerning the variables ρ, z;

d) there is an inequality

$$c_1 < c_2, \tag{2.8}$$

where

$$c_1 = \sup\left[V \middle| \rho < H_{00}, \quad |z_i| < H_{0i}, \quad i = \overline{1,m}, \quad t = 0\right], \tag{2.9}$$

$$c_2 = \inf\left[V \middle| |z_1| < H_1, \quad |z_i| < \infty, \quad i = \overline{2,m}, \quad \rho < \infty, \quad t \in I\right]. \tag{2.10}$$

Indeed, conditions a), b), c) are sufficient for the asymptotic stability of system (1.1) – (1.3) (Sirazetdinov, 1987), and the fulfillment of estimate (1.5) follows from inequalities $V(t) < V(0) \le c_1 < c_2$, which, according to conditions a), b), c), d) take place on any system solution (1.1) – (1.3) starting from region (1.4). But if $V < c_2$, then, by the constant c_2 definition, (1.5) holds.

Let the matrix $v(x)$ be definitely positive for any $x \in [0,1]$. Then the quadratic form V_1, taking into account the boundedness of the elements of the matrix $v(x)$, satisfies the conditions

$$\lambda_1 \rho \le V_1 \le \lambda_2 \rho, \quad \lambda_1, \lambda_2 - const > 0, \tag{2.11}$$

where λ_1, λ_2 are numbers that limit the characteristic numbers of the matrix $v(x)$ from below and above, respectively.

Suppose that the matrix Q is also definitely positive. Then the quadratic form V_2 satisfies the inequalities (Bayramov, 1995)

$$\frac{\Delta z_i^2}{\Delta_i} \le V_2 \le \sum_{i,j=1}^{m} |q_{ij}| |z_i| |z_j|, \quad i = \overline{1,m}, \tag{2.12}$$

where q_{ij} – elements of matrix Q, $\Delta = \det Q$, Δ_i – addition to i - that diagonal element Δ.

In accordance with (2.8) – (2.11) for the numbers c_1, c_2 we take

$$c_1 = \lambda_2 H_{00} + \sum_{i,j=1}^{m} |q_{ij}| H_{0i} H_{0j}, \quad c_2 = \frac{\Delta H_1^2}{\Delta_1}. \tag{2.13}$$

Condition (213) is written:

$$\lambda_2 H_{00} + \sum_{i,j=1}^{m} |q_{ij}| H_{0i} H_{0j} < \frac{\Delta H_1^2}{\Delta_1}. \tag{2.14}$$

Thus, conditions a), b), c), d) will be satisfied if Q, H are definitely positive, and the matrices $v(x)$, $w(x)$ are definitely positive with $x \in [0,1]$, i.e.

$$Q > 0, \quad H > 0; \quad v(x) > 0, \quad w(x) > 0, \quad x \in [0,1], \tag{2.15}$$

and there is an inequality (2.15).

2. Example. Consider the stability of the rotor type wind turbine with a vertical axis of rotation together with the load (generator, pump, etc.). The shaft transmitting the torque of the wind turbine to the load has a considerable length, so the problem is solved

taking into account the elasticity of this shaft (Bairamov & Mardamshin, 2008).

The equations of the dynamics of a wind turbine with a load and an elastic gear shaft in relative deviations from the nominal operating mode have the form (Bairamov et al., 2009).

$$\frac{dz}{dt}=kz+\frac{\partial\varphi(x,t)}{\partial x}_{|x=0},$$

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}=a\frac{\partial^2\varphi(x,t)}{\partial x^2},\quad x\in(0,1),$$

$$\frac{\partial\varphi(x,t)}{\partial x}_{|x=1}=-k_1\frac{\partial\varphi(x,t)}{\partial t}_{|x=1},\quad \frac{\partial\varphi(x,t)}{\partial t}_{|x=0}=k_2 z. \tag{3.1}$$

Here $x=\frac{y}{\lambda}$, $z=\frac{\omega-\omega_*}{\omega_*}$, $\varphi(x,t)=\frac{\psi(x,t)-\psi_*(x,t)}{\psi_{\max *}}$, $\psi_{\max *}=\frac{M_* l}{GI}$, $a=\frac{GI}{Jl^2}$, $k=\frac{1}{J_W}\left(\frac{\partial M}{\partial\omega}\right)_*$, $k_1=\frac{1}{GI}\left(\frac{\partial M_P}{\partial\omega}\right)_*$, $k_2=\frac{GI\omega_*}{lM_*}$, y are the coordinates of the cross-sections of the transmission shaft, ω is the angular speed of the wind turbine, $\psi(x,t)$, λ, J, GI is the absolute angle of rotation of the sections, length, running moment torsional rigidity of the transmission shaft, M, M_P – torques of the wind turbine and pump, J_W – moment of inertia of the wind turbine, $\psi_{\max}$ – maximum angle of rotation of the transmission shaft in the nominal mode, the sign ($*$) indicates the values of the values in the nominal operating mode of the unit, when $\omega_*=const$, $M=M_P=M_*=const$ and the transmission shaft and has a constant static deformation of $\partial\psi_*/\partial x=M_*/GI$.

Introducing the new variables $\varphi_1=\partial\varphi/\partial t$, $\varphi_2=\partial\varphi/\partial x$ and taking into account the integrability condition $\partial\varphi_2/\partial t=\partial\varphi_1/\partial x$, we write equations (3.1) in the form of system (1.1) – (1.3), where

$$A=\begin{Vmatrix}0 & a\\ 1 & 0\end{Vmatrix},\quad F_1=k,\quad F_2=\|0\quad 1\|,\quad \Gamma_1=\|1\quad 0\|,\quad \Gamma_2=k_2,\quad \Gamma_3=\|k_1\quad 1\|, \tag{3.2}$$

and the matrices A_0, B, B_0, C, C_0, D, D_0, F_3, Γ_4 are zero.

We construct functional (2.1), wherein this example we take $V_2=qz^2$, $q=const>0$. We write equation (2.2) and (2.5) in a scalar form. Given that in this case $P_1=P_2=0$, we get

$$v_{22}=av_{11}, \tag{3.3}$$

$$\frac{dv_{12}}{dx}=w_{11},\quad a\frac{dv_{12}}{dx}=w_{22},\quad a\frac{dv_{11}}{dx}=w_{12}, \tag{3.4}$$

where v_{ij}, w_{ij} are elements of matrices v, w.

Equations (2.3) imply the following boundary conditions for $x=0$ and $x=1$:

$$q=av_{11}(0),$$

$$v_{12}(0)=0,\quad (1+ak_1^2)v_{12}(1)=2ak_1v_{11}(1). \tag{3.5}$$

Put $w_{11}=1$, $w_{22}=a$, $w_{12}=0$.

Solving equations (3.2) under the boundary conditions (3.5), we obtain

$$v_{11}=\frac{(1+ak_1^2)}{2ak_1},\quad v_{12}=x, \tag{3.6}$$

and from equation (2.6) we find $H=2qk$.

The functional V (2.1) and its derivative dV/dt (2.4), by system (3.1), can be written in the form:

$$V=\int_0^1\left[\frac{1+ak_1^2}{2ak_1}\left(\varphi_1^2+a\varphi_2^2\right)+2x\varphi_1\varphi_2\right]dx,$$

$$\frac{dV}{dt}=-\int_0^1\left(\varphi_1^2+a\varphi_2^2\right)dx+\frac{k(1+ak_1^2)}{k_1}z^2. \tag{3.7}$$

Since $k_2>0$, from inequalities (2.13) we find the conditions for the asymptotic stability of the wind turbine, taking into account the elasticity of the transmission shaft and the expression for k in the form:

$$\left(\frac{\partial M}{\partial\omega}\right)_*<0,\quad 0<k_1<\frac{1}{\sqrt{a}}. \tag{3.8}$$

Comparing these conditions with the stability condition $\left(\frac{\partial M_P}{\partial\omega}\right)_*-\left(\frac{\partial M}{\partial\omega}\right)_*>0$ for a wind turbine with a rigid gear shaft, we see that the elasticity of the shaft narrows the stability region.

4 Summary

- Equations are developed for constructing Lyapunov functions in the form of a sum of ordinary and integral quadratic forms.
- The stability conditions for systems with distributed and concentrated parameters in the form of inequalities connecting the coefficients of the system and the condition for the accuracy of its operation are obtained, under which the deviations of some of the main state variables of the system from their calculated values remain within predetermined limits.
- The work has theoretical and practical value. The results can be used in the design and study of various complex engineering objects with distributed and concentrated parameters.

5 Conclusion

Using the Lyapunov function method, we study the stability and accuracy of the functioning of systems with distributed and concentrated parameters described by linear differential equations in partial and ordinary derivatives. The proposed approach is related to the idea of transforming high-order equations into a system of first-order equations in time and spatial coordinates and constructing Lyapunov functions for them in the form of a sum of ordinary and integral quadratic forms. Such an approach allows constructing Lyapunov functions constructively using specific equations and developing a universal methodology for studying the stability and accuracy of various systems with distributed and concentrated parameters.

Acknowledgments

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.

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Primary Paper Section: B

Secondary Paper Section: BA, BE