CONSERVATION LAWS FOR TWO DIMENSIONAL HIROTA-MAXWELL-BLOCH SYSTEMS

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Abstract: In this paper, we consider the two-dimensional Hirota-Maxwell-Bloch equation. Lax pairs are presented for this equation, conservation laws are obtained for the two-dimensional Hirota-Maxwell-Bloch equations.

Keywords: Hiroti and Maxwell-Bloch equations, Lax representation, integrability, conservation laws.

1 Introduction

Modern nonlinear science as a powerful subject explains all kinds of secrets in the problems of modern technology and science. The non-linear nature of real systems is considered fundamental to understanding most natural phenomena. Integrable systems are the main part of the theory of modern nonlinear science. One of the interesting integrable systems is the so-called one-dimensional Hirota-Maxwell-Bloch equations (HMBE). They describe the nonlinear dynamics of the propagation of a femtosecond pulse through a doped fiber. In this article, we will consider one of the two-dimensional integrable generalizations of one-dimensional HMBE, namely, two-dimensional HMBE. We note that many nonlinear partial differential equations (PDEs) admit an infinite number of conservation laws. Although most do not have a physical interpretation, these conservation laws play an important role in creating the complete integrability of PDEs.

2 Materials and Methods

The nonlinear Schrödinger equation is widely used in various fields of physics, for example, in nonlinear optics, plasma physics, superconductivity theory, and low-temperature physics. The structure of (1+1)-dimensional nonlinear Schrödinger equations is now very well studied. However, much is still not known about the properties of multidimensional nonlinear evolution equations.

Nonlinear equations have been the subject of research in various fields of nonlinear sciences. Nonlinear equations are often used to describe many problems in physics (heat flux and wave propagation phenomena), protein chemistry, quantum mechanics, plasma physics, wave propagation in shallow water, optical fibers, fluid mechanics, biology, solid-state physics. chemical kinematics, etc. It is widely known that the study of integrability and finding exact solutions of nonlinear equations are always one of the interesting topics in physics and mathematics. Over the past decade, the theory of various solutions has evolved in many different directions. Various nonlinear solutions, such as positons, solitons, and dromions, are presented for nonlinear integrable equations. Along with the development of the soliton theory, various powerful methods for working with nonlinear equations were developed, such as the inverse scattering transform [1], the Hirota bilinear method, and others. The theory of nonlinear partial differential equations has attracted much attention from researchers and is fundamentally connected with some major developments in the field of soliton theory. By a partial differential equation is meant an equation for a function of two or more variables containing at least one partial derivative of this function. Moreover, the function itself and independent variables may not be included in the equation explicitly. Any partial differential equation has an infinite number of solutions. Of greatest interest are solutions that satisfy the additional condition. These conditions are called boundary conditions and consist in specifying the behavior of the solution on some boundary line (surface) or in its immediate vicinity. From this point of view, the initial conditions are boundary conditions in time. Boundary conditions are used to select a particular solution from an infinite number of solutions. Almost any problem that describes a physical process and formulated in terms of partial differential equations includes boundary conditions.

2.1 Two-dimensional Hirota-Maxwell-Bloch equation and its reduction

One of the interesting integrable system is the so-called (1+1)-dimensional Hirota-Maxwell-Bloch system. It describes the nonlinear dynamics of femtosecond pulse propagation through doped fibre.

The two-dimensional Hirota-Maxwell-Bloch equations have the following form, (1-2)

$$iq_t + \varepsilon_1 q_{xy} + i\varepsilon_2 q_{xxy} - vq + i(wq)_x - 2ip = 0, \quad (1)$$

$$v_x + 2\varepsilon_1 \delta(|q|^2)_y - 2i\varepsilon_2 \delta(q^*_{xy} q - q^* q_{xy}) = 0, \quad (2)$$

$$w_x - 2\varepsilon_2 \delta(|q|^2)_y = 0, \quad (3)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (4)$$

$$\eta_x + \delta(q^* p + p^* q) = 0, \quad (5)$$

where q, p - complex functions, v, w, η - real-valued function. $\varepsilon_1, \varepsilon_2, \delta, \omega$ - real constants and $\delta = \pm 1$. The symbol * denotes complex pairing. This system is integrated by the inverse scattering method and admits the following integrable reductions: (2,3)

Case 1: $\varepsilon_1 = 1, \varepsilon_2 = 0$

$$iq_t + q_{xy} - vq - 2ip = 0, \quad (6)$$

$$v_x + 2\delta(|q|^2)_y = 0, \quad (7)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (8)$$

$$\eta_x + \delta(q^* p + p^* q) = 0, \quad (9)$$

We obtain the (2+1)-dimensional Schrödinger-Maxwell-Bloch equations when $\varepsilon_1 = 1, \varepsilon_2 = 0$ (2,4)

Case 2: $\varepsilon_1 = 0, \varepsilon_2 = 1$

In this case, when $\varepsilon_1 = 0, \varepsilon_2 = 1$ we get (2+1)-dimensional complex modified Korteweg-de-Frieze-Maxwell-Bloch equations: (2,5)

$$iq_t + iq_{xxy} - vq + i(wq)_x - 2ip = 0, \quad (10)$$

$$v_x - 2i\delta(q^*_{xy} q - q^* q_{xy}) = 0, \quad (11)$$

$$w_x - 2\delta(|q|^2)_y = 0, \quad (12)$$

$$p_x - 2i\omega p - 2\eta q = 0, \tag{13}$$

$$\eta_x + \delta(q^* p + p^* q) = 0, \tag{14}$$

Case 3: $\varepsilon_1 = 1, \varepsilon_2 = 1, p = 0, \eta = 0$

In the third case, when $\varepsilon_1 = 1, \varepsilon_2 = 1, p = 0, \eta = 0$ we get two-dimensional Hirota equations: (2,6)

$$iq_t + q_{xy} + iq_{xxy} - vq + i(wq)_x = 0, \tag{15}$$

$$v_x + 2\delta(|q|^2)_y - 2i\delta(q^*_{xy}q - q^* q_{xy}) = 0, \tag{16}$$

$$w_x - 2\varepsilon_2\delta(|q|^2)_y = 0, \tag{17}$$

In this paper, our goal is to find conservation laws for the two-dimensional Hirota-Maxwell-Bloch equations through the Lax representation.

3 Results and Discussion

3.1 Lax representation

The corresponding Lax representation is given as

$$\Psi_x = A\Psi, \tag{18}$$

$$\Psi_t = (2\varepsilon_1\lambda + 4\varepsilon_2\lambda^2)\Psi_y + B\Psi, \tag{19}$$

where A and B have the following form

$$A = -i\lambda\sigma_3 + A_0, \tag{20}$$

$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1}. \tag{21}$$

Here

$$B_1 = iw\sigma_3 + 2i\varepsilon_2\sigma_3 A_{0y}, \tag{22}$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{23}$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & i\varepsilon_1 q_y - \varepsilon_2 q_{xy} - wq \\ i\varepsilon_1 r_y + \varepsilon_2 r_{xy} + wr & 0 \end{pmatrix}, \tag{24}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix} \tag{25}$$

and $r = \delta q^*, \quad k = \delta p^*$, as well as $\delta = \pm 1$. The spectral parameter λ is detected as

$$\lambda_t = (2\varepsilon_1\lambda + 4\varepsilon_2\lambda^2)\lambda_y. \tag{26}$$

In this article, we restrict ourselves to the case $\delta = +1$ that corresponds to the attractive interaction. We note that by $y = x$ system (1)-(5) turns into one-dimensional HMBE. (7) This fact explains why we called system (1)-(5) two-dimensional HMBE.

4 Conservation laws

Knowing the laws of the action of forces on a system of particles and the state of a system of particles (coordinates and velocities of all particles) at a certain initial moment of time, it is possible to predict its further behavior using the equations of motion, that is, to find the state of the system at any moment in time. However, a detailed consideration of the behavior of the system using the equations of motion is often associated with great mathematical difficulties. And in those cases when the laws of action of forces are unknown, this approach is in principle impracticable. Therefore, the question arises: are there any general principles that would allow a different approach to solving the problem?

It turns out there are such principles. These are conservation laws. Conservation laws allow us to consider the general properties of motion without solving the equations of motion and detailed information on the development of processes in time. Conservation laws were established empirically, as a generalization of a huge number of experimental facts. In mechanics, three conservation laws matter: the conservation of energy, the conservation of momentum, the conservation of angular momentum. These laws are among those fundamental principles of physics, the importance of which is difficult to overestimate. Their role especially increased after it became clear that they go far beyond the framework of mechanics and represent universal laws of nature. In any case, still, not a single phenomenon has been discovered where these laws were violated.

Having opened up the possibility of a different approach to the consideration of various mechanical phenomena, conservation laws have become a powerful and effective research tool that physicists use every day. This crucial role of conservation laws as a research tool is due to the following reasons.

Conservation laws do not depend on the trajectories of movement, nor the nature of the acting forces. Therefore, they allow one to obtain some general and essential conclusions about the properties of various mechanical processes without going into a detailed discussion of them using the equations of motion.

Since conservation laws do not depend on the nature of the acting forces, they can be used even when the forces are unknown. In these cases, conservation laws are the only and indispensable research tool.

Even in cases where the forces are exactly known, conservation laws should be used in solving many problems of particle motion. Although all these problems can be solved using the equations of motion, the use of conservation laws very often allows us to obtain a solution most simply, saving us from tedious mathematical calculations. Therefore, when solving new problems, it is usually customary to adhere to the following order: first of all, the conservation laws are applied, and only after making sure that this is not enough, the equations of motion are used to solve the problem.

Conservation laws are certain laws according to which some physical quantities are preserved without changing with time in certain interactions. Conservation laws play an important role in understanding the mechanisms of interaction of particles, their formation, and decay. Conservation laws determine the selection rules, according to which processes with particles leading to violation of conservation laws can occur in certain types of interactions. In addition to the conservation laws in force in the macrocosm, new conservation laws have been discovered in the physics of the microworld that explain the observed experimental laws. (8-10)

Some of the conservation laws are always satisfied under any conditions (for example, the laws of conservation of energy, momentum, angular momentum, electric charge) or, in any case, processes that contradict these laws have never been observed. (11) Other laws are approximate and valid only under certain conditions (for example, the law of conservation of parity is valid for strong and electromagnetic interactions, but is violated with weak interactions).

Conservation laws are the result of a generalization of experimental observations. Some of them were discovered as a result of the fact that the reactions or decays allowed by all previously known conservation laws were not observed or were strongly suppressed. So, the laws of conservation of baryonic, lepton charges, strangeness, charm, and others were discovered. Conservation laws can be considered as one of the integrable properties for nonlinear evolution equations. (12-15) Recently, several methods have been proposed for deriving conservation laws, for example, through the Lax representation, (16) the Bäcklund transform, (17) formal solutions of the eigenfunctions, (17-18) the scattering problem, (16-17) and the quasi-differential operator based on the theory Sato. (19-20)

Now we find the conservation laws for the system (1)-(5). From $\Gamma = \Psi_2/\Psi_1$ and $\overline{w} = iq\Gamma$ (21) Riccati type equation can be obtained through the Lax representation (18):

$$\bar{w}_x - \frac{q_x}{q}\bar{w} + i|q|^2 - 2i\lambda\bar{w} - i\bar{w}^2 = 0. \tag{27}$$

We rewrite $\bar{w}$ into the form of formal power series in regard to $1/\lambda$,

$$\bar{w} = \sum_{n=1}^{\infty}\frac{\bar{w}_n}{\lambda^n}. \tag{28}$$

Substituting series (28) into the Riccati equation (27) and equating the expressions with the same powers λ, we obtain:

$$\bar{w}_1 = \frac{1}{2}|q|^2, \tag{29}$$

$$\bar{w}_2 = -\frac{i}{4}qq_x^*, \tag{30}$$

$$\bar{w}_3 = -\frac{1}{8}\left(|q|^4 + qq_{xx}^*\right) \tag{31}$$

Substituting the above expressions into the compatibility condition $\left(\frac{\Psi_{1t}}{\Psi_1}\right)_x = \left(\frac{\Psi_{1x}}{\Psi_1}\right)_t$, we obtain an infinite number of conservation laws for the system (1)-(5):

$$\frac{\partial \rho_i}{\partial t} = \frac{\partial J_i}{\partial x},\ i = 1,2,3.... \tag{32}$$

In accordance with (12,21-22), ρ_i and J_i $(i = 1,2,...)$ are the conserved densities and fluxes, respectively. The first three conservation laws that describe energy, momentum, and the Hamiltonian have the following form:

$$\rho_1 = -\frac{i}{2}|q|^2, \tag{33}$$

$$\rho_2 = -\frac{i}{2}w|q|^2 - \frac{1}{4}qq_x^*, \tag{34}$$

$$\rho_3 = -\frac{i}{8}|q|^4 - \frac{1}{4}wqq_x^* - \frac{i}{8}qq_{xx}^*. \tag{35}$$

In deriving (33)-(35), we did not use (19); therefore, these expressions coincide for any equations solvable by (18).

$$J_1 = 8i\omega^3\eta q + 4\omega^2 pq^* + 2i\omega pq_x^* - pq^*|q|^2 - pq_{xx}^*, \tag{36}$$

$$J_2 = 4i\omega^3\varepsilon_2 qq_x^* + 4w^3\varepsilon_1 q_y q^* + 4i\omega^3\varepsilon_2 q_{xy} q^* + 4i\omega^3 w|q|^2 - \\ -4\omega^2 pq^* - 2i\omega pq_x^* + pq^*|q|^2 + pq_{xx}^*, \tag{37}$$

$$J_3 = 2w^2\varepsilon_2 q_{xy} q_x^* - 2w^2\varepsilon_2|q|^4 - 2w^2\varepsilon_2 qq_{xx}^* - 2iw^2\varepsilon_1 q_y q_x^* + \\ +2\omega^2 wqq_x^* + 2i\omega pq_x^* - pq^*|q|^2 - pq_{xx}^*. \tag{38}$$

4 Conclusion

The Hirota and Maxwell-Bloch equations are well-known partial differential equations that provide a successful model in nonlinear optical theory. In this paper, for the first time, conservation laws were found for the two-dimensional Hirota-Maxwell-Bloch equations with the corresponding Lax representation, which play an important role in creating complete integrability of the PDE.

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Primary Paper Section: B

Secondary Paper Section: BA